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Hybrid Metaheuristics for the Clustered Vehicle Routing Problem

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Abstract

The Clustered Vehicle Routing Problem (CluVRP) is a variant of the Capacitated Vehicle Routing Problem in which customers are grouped into clusters. Each cluster has to be visited once, and a vehicle entering a cluster cannot leave it until all customers have been visited. This article presents two alternative hybrid metaheuristic algorithms for the CluVRP. The first algorithm is based on an Iterated Local Search algorithm, in which only feasible solutions are explored and problem-specific local search moves are utilized. The second algorithm is a Hybrid Genetic Search, for which the shortest Hamiltonian path between each pair of vertices within each cluster should be precomputed. Using this information, a sequence of clusters can be used as a solution representation and large neighborhoods can be efficiently explored, by means of bi-directional dynamic programming, sequence concatenation, and appropriate data structures. Extensive computational experiments are performed on benchmark instances from the literature, as well as new large scale instances. Recommendations on the choice of algorithm are provided, based on average cluster size.

Keywords: Clustered Vehicle Routing, Iterated Local Search, Hybrid Genetic algorithm, Large Neighborhoods, Shortest Path

1. Introduction

This paper addresses the *Clustered Vehicle Routing Problem* (CluVRP), which has been introduced by Sevaux and Sörensen (2008). The CluVRP is defined over an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where the vertex 0 is the *depot* and any other vertex $i \in \mathcal{V} \setminus \{0\}$ is a customer with demand $q_i > 0$. A fleet of m vehicles, each with capacity Q , is stationed at the depot. The

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set of customers is partitioned into N disjoint and nonempty subsets called *clusters*, such that $\mathcal{V} = V_1 \cup \dots \cup V_N$. The customers in each cluster have to be visited consecutively, that is, the vehicle visiting a customer in the cluster cannot leave the cluster until all the other customers in the cluster have been visited. Each edge $(i, j) \in \mathcal{E}$ is associated with a travel cost c_{ij} , and the objective is to minimize the total travel cost. The CluVRP is a generalization of the Capacitated Vehicle Routing Problem (CVRP, c.f. book of Toth and Vigo 2002), obtained when each cluster contains a single vertex, and of the Clustered Traveling Salesman Problem (CluTSP, Chisman, 1975), obtained when $m = 1$. The CVRP and the CluTSP are both \mathcal{NP} -Hard, and so is the CluVRP.

Sevaux and Sörensen (2008) introduced the CluVRP in the context of a real-world application where containers are employed to carry goods. The customers expecting parcels in the same container form a cluster, because the courier has to deliver the content of a whole container before handling another container. Clusters also arise in applications involving passenger transportation, where passengers prefer to travel with friends or neighbors (as in the transportation of elderly to recreation centres). Gated communities (residential or industrial areas enclosed in walled enclaves for safety and protection reasons) provide another natural example of clusters. The customers within a gated community are likely to be visited by a single vehicle in a sequence, otherwise the vehicles have to spend additional time for the security controls at the gates.

Clusters can thus be imposed by the geography, the nature of the application, as well as by practitioners aiming to achieve *compact* and easy-to-implement routing solutions. Clustered routes allow drivers to be assigned to areas (i.e., certain streets or postcodes) and allow the development of familiarity, which makes their task easier. In addition, clustered routes have significantly less overlaps. In several cases, the additional routing costs due to cluster constraints are compensated by the ease of implementation and the enhanced driver familiarity.

The literature on the CluVRP is quite limited as of the time of this writing. Sörensen et al. (2008) and Sevaux and Sörensen (2008) presented an integer programming formulation capable of finding the best Hamiltonian path between each pair of vertices in each cluster. Barthélemy et al. (2010) suggested to adapt CVRP algorithms to the CluVRP by including a large positive term M to the cost of the edges between clusters and a cluster and the depot. The CluVRP is solved as a CVRP by means of the algorithm of Clarke and Wright (1964) followed by 2-OPT moves and Simulated Annealing (SA). The authors also suggested to dynamically set the penalty M , but observed that the M term interferes with the Boltzmann acceptance criterion of the SA and leads to erratic performance. Computational results were not reported in this initial paper.

Pop et al. (2012) described the directed CluVRP as an extension of the *Generalized Vehicle Routing Problem* (GVRP, Ghiani and Improta, 2000). The authors adapted two polynomial-sized formulations for the GVRP to the directed CluVRP, but again no computational results were reported. Recently, Battarra et al. (2014) proposed exact algorithms for the CluVRP and provided a set of benchmark instances with up to 481 vertices. The best performing algorithm

relies on a preprocessing scheme, in which the best Hamiltonian path is precomputed for each pair of endpoints in each cluster. This allows for selecting a pair of endpoints in each cluster rather than the whole path, relegating some of the problem complexity in the preprocessing scheme. The resulting minimum cost Hamiltonian path problems are reduced to instances of the Traveling Salesman Problem (TSP) and optimally solved with Concorde (Applegate et al., 2001). CluVRP instances of much larger size than the corresponding CVRP instances were optimally solved, thus highlighting the advantage of acknowledging the presence of clusters.

In this paper, we introduce hybrid adaptations of state-of-the-art CVRP metaheuristics for the CluVRP. Rather than rediscovering well-known metaheuristic concepts, we exploit the current knowledge on iterated local search and hybrid genetic algorithms (Subramanian, 2012; Vidal et al., 2014a) and focus our attention on developing efficient problem-tailored neighborhood searches and effectively embedding them into these metaheuristic frameworks. The proposed neighborhood searches aim at 1) better exploiting clustering constraints by means of pruning techniques, 2) exploring larger neighborhoods by means of dynamic programming, 3) reducing the computational time by means of re-optimization, bi-directional search, and data structures. Finally, these experiments lead to further insights on which type of metaheuristic to use for different instance sizes and cluster characteristics.

The remainder of the paper is organized as follows. Section 2 introduces the challenges related to the CluVRP. Sections 3 and 4 describe the proposed metaheuristics and efficient neighborhood-search strategies, whereas Section 5 discusses our computational results. Conclusions are drawn in Section 6, and further avenues of research are discussed.

2. Motivation

Battarra et al. (2014) showed that exact algorithms are capable of solving relatively large CluVRP instances. However, the CPU times remain prohibitively long for large-scale or real time applications. In this paper, we exploit the properties of the CluVRP to develop specialized hybrid metaheuristics that take advantage of cluster constraints. Solution quality is assessed by a comparison with exact solutions whenever possible, and among metaheuristics when it is not.

Two recent and successful metaheuristic frameworks are used in this work. The Iterated Local Search (ILS) algorithm of Subramanian (2012) is simple and flexible, combining the intensification strength of Local Search (LS) operators and effective diversification through perturbation operators. It proved to be remarkably efficient for many variants of the Vehicle Routing Problem (VRP), including the VRP with Simultaneous Pickup and Delivery (Subramanian et al., 2010), the Heterogeneous VRP (Penna et al., 2013), the Minimum Latency Problem (Silva et al., 2012), and the TSP with Mixed Pickup and Delivery (Subramanian and Battarra, 2013). The success of ILS is due to an intelligent design of intensification and diversification neighbourhoods, as well as their random exploration. This latter component allows for extra diversity, and leads to high

quality solutions, even when applied to other problems such as scheduling (Subramanian et al., 2014).

ILS explores only feasible solutions, and allows for testing the M approach suggested by Barthélemy et al. (2010) without possible interferences between M and penalties applied to infeasible solutions. As mentioned in the introduction, the M approach consists of including a large positive term to all those edges that are connecting clusters and connecting the depot to the clusters. Any CVRP algorithm in which the M is chosen to be large enough returns a CluVRP solution in which the number of penalized edges is minimized, therefore a solution in which the cluster constraint is satisfied. Note that the number of edges connecting clusters or connecting the depot to a cluster is $m + N$ and their penalization can be easily deducted from the solution cost.

One drawback of this transformation is that most VRP neighborhoods consider moves of one or two vertices. These neighborhoods can often not relocate complete clusters, and thus many moves appear largely deteriorating due to M penalties, significantly inhibiting the progress towards higher quality solutions. As shown in this paper, ILS can partly overcome this issue by means of perturbation moves. However, as demonstrated by our computational results, a more clever application of the framework specific to the CluVRP considering relocating and exchanges of whole clusters and intra-cluster improvements produces solutions of comparable quality in considerably less CPU time. In the next section, we describe the ILS and these hybrid algorithms in more details.

The Unified Hybrid Genetic Search (UHGS) currently obtains the best known solutions for more than 30 variants of the CVRP and represents the state-of-the-art among hybrid metaheuristics for VRPs. More precisely, the algorithm successfully solves problems with diverse attributes, such as multiple depots and periods (Vidal et al., 2012), time windows and vehicle-site dependencies (Vidal et al., 2013a), hours-of-service-regulations for various countries (Goel and Vidal, 2013), soft, multiple, and general time windows, backhauls, asymmetric, cumulative and load-dependent costs, simultaneous pickup and delivery, fleet mix, time dependency and service site choice (Vidal et al., 2014a), and prize-collecting problems (Vidal et al., 2014c), among others. It has been recently demonstrated that several combinatorial decisions, such as customer selections or depot placement, can be relegated directly at the level of cost and route evaluations, allowing to always rely on the same metaheuristic and local search framework while exploring large neighborhoods in polynomial or pseudo-polynomial time (Vidal et al., 2014b,c).

Our UHGS implementation is based on the assumption that the costs of the optimal Hamiltonian paths among vertices in the same cluster can be efficiently precomputed as in Battarra et al. (2014). Once these paths and their costs are known, an effective route representation as an ordered sequence of clusters can be adopted, and a fast shortest path-based algorithm then transforms this solution representation into the corresponding optimal sequence of customers, which will be explained in detail in Section 4. This approach drastically reduces the size of the search

space of the UHGS method, which optimizes the assignment and sequencing of $\mathcal{O}(N)$ clusters instead of $\mathcal{O}(n)$ customers.

Our computational experiments allow to quantify the trade-off between adopting the preprocessing scheme to compute the Hamiltonian paths, which requires the solution of $\sum_{i=1,\dots,N} |V_i| \times (|V_i| - 1)$ TSP instances and searching in the space of clusters with UHGS, or working in the space of vertices with a well-designed ILS. As long as the average size of the clusters is not high, the computational burden of the preprocessing is not prohibitive, but is observed to become significant when the cluster size increases. On the other hand, UHGS is much faster (ignoring the preprocessing time) and obtains higher quality solutions. Through our computational experiments, we aim at identifying a critical cluster size that makes an approach with cluster-based solution representation more desirable than an approach using vertex-based representation.

3. The ILS metaheuristic

ILS is a well-known metaheuristic framework that iteratively alternate stages of local search (intensification) and perturbation moves (diversification). The interested reader is referred to Lourenço et al. (2003) for a detailed description of the methodology, whereas the structure of a typical ILS algorithm is presented in Algorithm 1.

The algorithm starts by generating an initial solution s_0 ; this solution is then improved by means of local search ($LocalSearch(s_0)$) and a local optimal solution s^* is obtained. Perturbation moves are applied to s^* , generating a new solution s' , which in turn is improved by means of local search, generating the solution $s^{*'}.$ The algorithm updates s^* if an acceptance criterion is met. [A typical stopping criterion is to interrupt the execution of the algorithm if no improvement is obtained after \$n_I\$ consecutive iterations.](#)

Algorithm 1 Iterated Local Search

```

1: Procedure ILS:
2:  $s_0 \leftarrow \text{GenerateInitialSolution};$ 
3:  $s^* \leftarrow \text{LocalSearch}(s_0);$ 
4: While Stopping criterion is not met
5:    $s' \leftarrow \text{Perturb}(s^*, \text{history});$ 
6:    $s^{*'} \leftarrow \text{LocalSearch}(s');$ 
7:    $s^* \leftarrow \text{AcceptanceCriterion}(s^*, s^{*'}, \text{history});$ 
8: end ILS;
```

The ILS of Subramanian (2012) is a multi-start heuristic which returns the best solution after n_R restarts (i.e., the Algorithm 1 is executed n_R times). The initial solution is generated using a *parallel cheapest insertion heuristic*. Classical VRP/TSP neighborhoods are explored during the local search phase and the perturbation operator consists of multiple shift and swaps based

moves selected at random. The neighborhood structures adopted in Subramanian (2012) are inter-route moves (RELOCATE, RELOCATE2, SWAP, SWAP(2,1), SWAP(2,2), 2-OPT*), as well as intra-route moves (RELOCATE, OR-OPT2, OR-OPT3, 2-OPT and SWAP). Detailed descriptions of these families of neighborhoods can be found in Subramanian (2012) and Vidal et al. (2013b). [Inter-route LS neighborhoods are considered one by one in random order and, whenever an improving solution is found, intra-route LS operators are applied, also in random order, to this solution.](#)

As previously mentioned, the algorithm of Subramanian (2012) can be used for solving the CluVRP by applying suitable penalties to edges between clusters and between clusters and the depot. Although simple, this straightforward adaptation has two main drawbacks: (i) most of the local search moves violate the cluster constraint, leading to high penalties, and consuming a large part of the CPU time; and (ii) many promising moves that relocate full clusters are not included in the neighborhoods, thus reducing the intensification capabilities of the LS. ILS was therefore adapted to better take advantage of clusters. In what follows, we denote this adaptation as ILS-Clu.

The ILS-Clu is a hybrid algorithm built upon the structure of ILS. Large neighborhoods proved to be very effective in solving VRP variants, however, to identify promising moves can be a difficult task that is usually left for a large part to randomization (e.g., in Adaptive Large Neighborhood Search, Pisinger and Ropke, 2007). In contrast, the CluVRP structure enables to apply moves to relevant sets of customers. Thus, the LS phase of ILS-Clu explores moves on different levels: among clusters, among edges connecting clusters or clusters with the depot, and within each cluster. This mechanism enables to explore a sufficiently large variety of moves while significantly reducing CPU time. As in ILS, the initial solution is generated using a *parallel cheapest insertion heuristic*. Iteratively, a randomly selected customer is inserted with minimum cost, either between customers from the same cluster, or between two clusters.

Four types of LS procedures are used in ILS-Clu: “InterRouteSearch_C”, “IntraRouteSearch_C”, “IntraClusterSearch” and “IntraClusterRestrictedSearch”. The former two modify the sequence of clusters in the routes without changing the Hamiltonian paths and their endpoints in each cluster, whereas the latter two optimize the sequence of customers within each cluster. [The set of neighborhoods used during “InterRouteSearch_C” contains](#) the same inter-route neighborhoods as ILS – previously described – but moves are applied on clusters instead of single deliveries. The intra-route neighborhoods of “IntraRouteSearch_C” follow the same rationale.

The LS procedures “IntraClusterSearch” and “IntraClusterRestrictedSearch” rely on the RELOCATE, 2-OPT, and SWAP neighborhoods. “IntraClusterRestrictedSearch” explores only a linear number of LS-moves, more precisely those involving at least one endpoint customer in the cluster, whereas “IntraClusterSearch” explores all moves within a cluster. Algorithm 2 displays the main structure of ILS-Clu, and highlights the differences with ILS.

LEFT TO DO

Both ILS and ILS-Clu apply a perturbation mechanism after each LS stage, which consists

Algorithm 2 Local search of ILS and ILS-Clu

Local Search of ILS:NL \leftarrow set of InterRouteSearch neighborhoods;**While** NL $\neq \emptyset$ Choose randomly Neighborhood \in NL; Find best s' of $s \in$ Neighborhood; **If** $f(s') < f(s)$ **then** $s \leftarrow s'$; $s \leftarrow$ IntraRouteSearch(s);

Update NL;

else

Remove Neighborhood from NL;

return s ;**end.**

Local Search of ILS-Clu: $NL_C \leftarrow$ set of InterRouteSearch_C neighborhoods;**While** $NL_C \neq \emptyset$ Choose randomly Neighborhood $\in NL_C$; Find best s' of $s \in$ Neighborhood; **If** $f(s') < f(s)$ **then** $s' \leftarrow$ IntraClusterRestrictedSearch(s'); $\bar{s} \leftarrow$ IntraRouteSearch_C(s'); **If** $f(\bar{s}) < f(s')$ **then** $s \leftarrow$ IntraClusterRestrictedSearch(\bar{s}); **else** $s \leftarrow \bar{s}$; Update NL_C ; **else** Remove Neighborhood from NL_C ; $s \leftarrow$ IntraClusterSearch(s);**return** s ;**end.**

of one or two randomly selected SHIFT(1,1) or SWAP moves. In ILS, SHIFT(1,1) relocates a random customer from its route r to a random position in another route r' , and simultaneously relocates a random customer from r' to a random position in r . The same process is applied in ILS-Clu but considering clusters instead of single customers. Moreover, in ILS, SWAP exchanges two customers from different routes, whereas in ILS-Clu the exchange involves two clusters of the same route. **LEFT TO DO**

4. The UHGS metaheuristic

UHGS is a successful framework capable of producing high quality solutions for many VRP variants. It is a hybrid algorithm, where the diversification strength of a Genetic Algorithm (GA) is combined with the improvement capabilities of local search. One main challenge in the design of a hybrid genetic algorithm is to achieve a good balance between intensification and diversification while controlling the use of computationally intensive local search procedures. This balance is usually achieved by selecting a suitable initial population, crossover operators, mutation, and selection mechanisms. The variety of design choices and the tuning of a multitude of parameters often inhibit the flexibility of the GAs. In fact, most of the previous attempts in the literature focused on the design of problem-specific operators, failing to lead to general algorithms and frequently resulting in a large number of parameters to be tuned. UHGS (Vidal et al., 2014a) managed to overcome most of these drawbacks by adopting the following strategies.

4.1. General UHGS methodology

UHGS evolves a population of individuals representing problem solutions, by means of selection, crossover and education operators. Note that the operator *education* involves a complete local-search procedure aimed at improving the solutions rather than a randomized mutation. The population is managed to contain between μ^{MIN} and $\mu^{\text{MIN}} + \mu^{\text{GEN}}$ individuals, by pruning μ^{GEN} individuals whenever the maximum size is attained. The method is run until It_{max} individuals have been successively created without improvement of the best solution.

UHGS achieves a fine balance between intensification and diversification by means of a bi-criteria evaluation of solutions. The first criterion is the contribution of a solution to the population diversity, which is measured as the Hamming distance of the solution to the closest solutions in the population. The second criterion is the objective value. Solutions are ranked with respect to both criteria, and the sum of the ranks provides a “biased fitness” (Vidal et al., 2014a), used for both parents selection and survivors selection when the maximum population size is attained. To deal with tightly constrained problems, linearly penalized route-constraint violations – capacity or distance – are included in the objective. Penalty coefficients are dynamically adjusted to ensure a target ratio of feasible solutions during the search; infeasible solutions are managed in a secondary population.

During crossover, the whole solution is represented as a giant tour visiting all customers once, without intermediate depot trips. As such, a simple ordered crossover that works on permutations can be used. The optimal splitting of the giant tour into separate routes is performed optimally in polynomial time as a shortest path subproblem on an auxiliary graph (Prins, 2004). This process is known to be widely applicable in a unified manner to many vehicle routing variants as long as it is possible to perform separate efficient route evaluations to compute the cost of edges in the auxiliary graph (Vidal, 2013). Finally, UHGS relies on local search to improve every new offspring solution generated during the search. The LS operators used in UHGS are 2-OPT, 2-OPT*, CROSS and I-CROSS (Vidal et al., 2014a). The last two neighborhoods are limited to exchanges, with possible inversions, of up to two customers.

Local search is usually the bottleneck of most advanced metaheuristics for vehicle routing variants, and thus efficient evaluations of routes generated by the neighborhoods are critical for the overall algorithm’s performance. When additional attributes (constraints, objectives or decisions) are considered, these route evaluations may be time consuming if implemented in a straightforward manner. To improve this process, UHGS relies on auxiliary data structures that collect partial information on any sub-sequence of consecutive customers in the incumbent solution. This information is then used for efficiently evaluating the cost and feasibility of new routes generated by local search moves since any such move can be seen as a recombination of subsequences of consecutive customers from the incumbent solution.

A simple illustration of this concept is now given. Consider a CVRP solution with two routes: $r_1 = (0, 1, 2, 3, 4, 5, 0)$ and $r_2 = (0, 6, 7, 8, 9, 0)$. To efficiently evaluate the capacity constraints,

the partial load $Q(\pi)$ for any sub-sequence π of the incumbent solution is preprocessed prior to move evaluations. An inter-route 2-OPT* move breaking the edges (3, 4) and (7, 8) leads to two new routes $r'_1 = (0, 1, 2, 3, 8, 9, 0)$ and $r'_2 = (0, 6, 7, 4, 5, 0)$ requiring cost and load feasibility evaluations. Loads $Q(\pi_{0 \rightarrow 3})$, $Q(\pi_{4 \rightarrow 0})$, $Q(\pi_{0 \rightarrow 7})$ and $Q(\pi_{8 \rightarrow 0})$ are known for sequences (0, 1, 2, 3), (4, 5, 0), (0, 6, 7) and (8, 9, 0), respectively. Denoting \oplus as the concatenation operator, we have $Q(r'_1) = Q(\pi_{0 \rightarrow 3} \oplus \pi_{8 \rightarrow 0}) = Q(\pi_{0 \rightarrow 3}) + Q(\pi_{8 \rightarrow 0})$, and in the same way $Q(r'_2) = Q(\pi_{0 \rightarrow 7}) + Q(\pi_{4 \rightarrow 0})$. Load constraints can thus be checked in $\mathcal{O}(1)$ operations, independently of the number of customers in the route. Otherwise, a straightforward approach sweeping through the new route and cumulating the demands would take a number of operations proportional to $\mathcal{O}(n)$. This type of route evaluation is referred to as move evaluation *by concatenation* in Vidal et al. (2014a), and can be applied to a wide range of resources (load, distance, time), constraints and objectives.

4.2. Application to the CluVRP

Our application of UHGS to the CluVRP relies on two contributions: a route representation based on an ordered sequence of clusters to reduce the search space, and efficient route evaluation procedures using concatenations to evaluate the cost of a route assimilated to a sequence of clusters. Thus, UHGS is applied on sequences of clusters, and the optimal path within the clusters is determined implicitly during move and route evaluations. This leads to an implicit structural problem decomposition, considering only a VRP of a size proportional to the number of clusters $N < n$, and relegating difficult combinatorial decisions on path selections within clusters at the level of route evaluations. These methodological elements can be easily integrated into the UHGS framework, and it was possible to use the original UHGS implementation with the sole addition of a new route-evaluation operator.

The method relies on the fact that in any cluster V_k , the cost \hat{c}_{ij} of the best Hamiltonian path between customer $i \in V_k$ and customer $j \in V_k$ that services all other customers in $V_k \setminus \{i, j\}$ has been preprocessed (Battarra et al., 2014). Using this information, it is possible to obtain from a route represented as a sequence of clusters the best sequence of visits to customers in polynomial time by finding a shortest path in the auxiliary graph \mathcal{G}' illustrated in Figure 1.

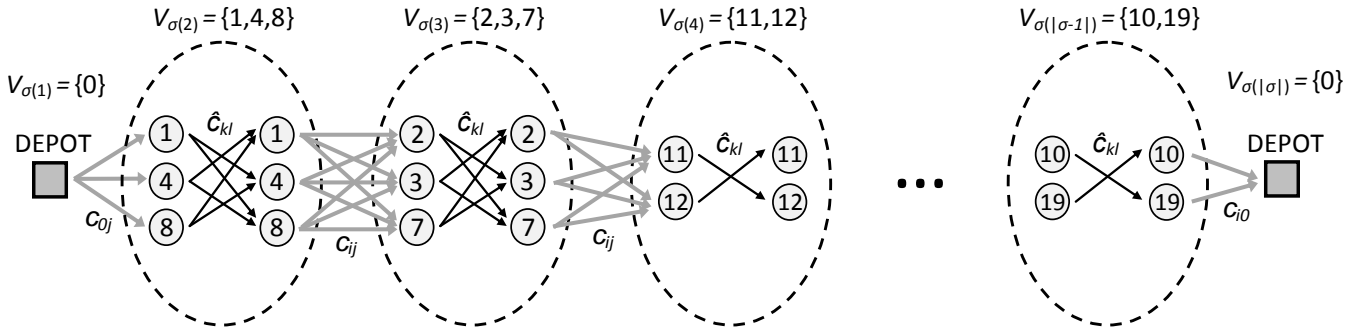


Figure 1: Route representation in UHGS

In Figure 1, black lines correspond to precomputed Hamiltonian paths within clusters. For each cluster in the route, a set containing two copies of each node is generated. Pairs of node copies are connected by an arc (k, l) . The cost of this arc is set to the cost of the shortest Hamiltonian path \hat{c}_{kl} in the cluster between k and l . The depot is then connected to the first copy of each node in the first cluster, and the second copies of the nodes are connected to the first node copies of the next cluster, and so on. The cost associated to these arcs (in gray in the figure) is the travel distance between the endpoints. A similar shortest path subproblem was previously used for the GVRP by Pop et al. (2013) and Vidal (2013).

A straightforward application of this technique leads to route evaluations in $\mathcal{O}(NB^2)$ operations, where B is the maximum number of customers in a cluster. These evaluations are computationally expensive. Another contribution of this work is to show that efficient procedures based on preprocessing and concatenations allow for performing each move evaluation in amortized $\mathcal{O}(B^2)$ operations, thus only depending on the square of the cluster size. These more efficient evaluation procedures are now described.

For ease of notation, define $\lambda_i = |V_i|$ for any cluster V_i . In addition to the Hamiltonian paths within clusters, which are pre-processed a single time before starting the method, UHGS now pre-processes some information on each subsequence σ of consecutive clusters in each current solution. This information is updated whenever a solution change, e.g. a local search move, is applied. For any sequence $\sigma = (\sigma(1), \dots, \sigma(|\sigma|))$ of clusters, the method computes for any i^{th} customer of the cluster $\sigma(1)$, and any j^{th} customer of the cluster $\sigma(|\sigma|)$, the shortest path $S(\sigma)[i, j]$ inside the sequence of clusters connecting i and j . These values can be computed as follows.

First, let us assume a subsequence $\bar{\sigma} = (V_k)$ containing a single cluster. If the cluster is restricted to a single customer, i.e., $V_k = \{v_i\}$, then $S(\bar{\sigma})[i, i] = 0$. Otherwise, $S(\bar{\sigma})[i, j]$ is given in Equation (1), in which \hat{c}_{ij} represents the distance of the best Hamiltonian path connecting i and j in the cluster V_k .

$$S(\bar{\sigma})[i, j] = \begin{cases} +\infty & \text{if } i = j \\ \hat{c}_{ij} & \text{if } i \neq j \end{cases} \quad \text{for } i \in \{1, \dots, \lambda_k\} \text{ and } j \in \{1, \dots, \lambda_k\}, \quad (1)$$

Any longer subsequence can be viewed as a concatenation of shorter subsequences. Equation (2) enables to evaluate $S(\sigma)$ on a new sub-sequence resulting of the concatenation of a pair of subsequences σ_1 and σ_2 , by induction on the concatenation operation \oplus . It is a direct application of the Floyd-Warshall algorithm.

$$S(\sigma_1 \oplus \sigma_2)[i, j] = \min_{1 \leq x \leq \lambda_{\sigma_1(|\sigma_1|)}, 1 \leq y \leq \lambda_{\sigma_2(1)}} S(\sigma_1)[i, x] + c_{xy} + S(\sigma_2)[y, j], \quad (2)$$

for $i \in \{1, \dots, \lambda_{\sigma_1(1)}\}$ and $j \in \{1, \dots, \lambda_{\sigma_2(|\sigma_2|)}\}$

Equation (2) can be used to perform preprocessing on all subsequences of clusters in the

current solution by iteratively appending a sequence made of a single cluster at the end. The same equation is then applied during neighborhood exploration to compute the cost of routes resulting from local moves. These routes can be viewed as a concatenation of a bounded number of subsequences from the current solution (Vidal, 2013). Figure 2 illustrates this pre-processing and move-evaluations mechanism. We consider the relocation of a subsequence of clusters σ_2 into a route originally made of $\sigma_1 \oplus \sigma_3$. The new route is thus a concatenation of the three sequences $\sigma_1 \oplus \sigma_2 \oplus \sigma_3$. The original shortest path problem is depicted on the top of the figure. Since pre-processing has been done on subsequences from the incumbent solution, the sets of shortest paths $S(\sigma_1)[i, j]$, $S(\sigma_2)[i, j]$ and $S(\sigma_3)[i, j]$ are known. They describe the shortest paths between any origin node i in the first cluster of the sequence, and any destination node j in the last cluster of the sequence. As a consequence, these known paths can be substituted in the graph, leading to an equivalent shortest path problem over a smaller graph, illustrated in the bottom of the figure. The number of nodes and arcs in this new graph does not depend anymore on the number of clusters in the route, but only on the number of concatenations, thus resulting in a reduced computational complexity.

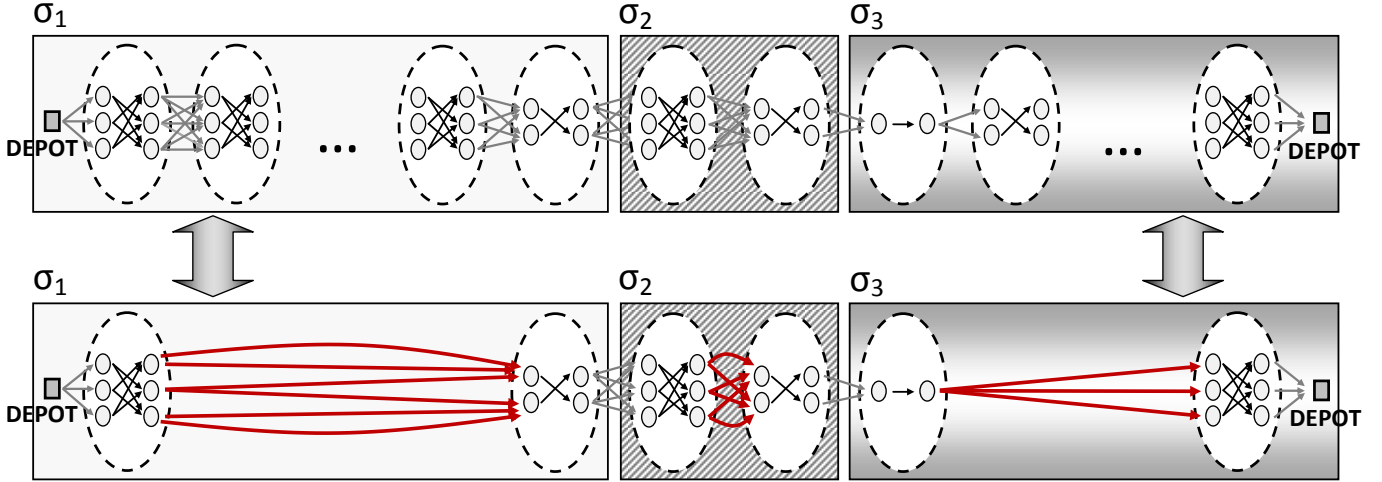


Figure 2: Using preprocessed information on subsequences

Proposition 1. *Using the proposed preprocessing, the amortized complexity of move evaluations, for classic VRP neighborhoods such as RELOCATE, SWAP, 2-OPT, 2-OPT*, is $\mathcal{O}(B^2)$.*

Proof. First, from the current incumbent solution, the preprocessing phase requires computing the shortest paths between each pair of nodes, for each route. For each route, the graph \mathcal{G}' is directed and acyclic. Equation (2) is applied iteratively, in lexicographic order starting from any cluster σ_i , $i \in \{1, \dots, |\sigma|\}$ and iteratively applied to σ_j for $j \in \{i + 1, \dots, |\sigma|\}$ to produce all shortest paths. This equation is thus used $\mathcal{O}(N^2)$ times to perform a complete preprocessing on all routes. Each evaluation of this expression requires $\mathcal{O}(B^2)$ time. The total effort for the

preprocessing phase is $\mathcal{O}(N^2B^2)$.

After preprocessing, a local search using classic VRP neighborhoods is performed. Any move based on less than k edge exchanges can be assimilated to a recombination of up to $k + 1$ subsequences of consecutive clusters. This is the case for the mentioned neighborhoods with $k \leq 4$. Thus, each move evaluation is performed with a bounded number of calls to Equation (2), in $\mathcal{O}(B^2)$ elementary operations. The size S of each neighborhood is quadratic in the number of clusters (e.g. swapping any cluster i with cluster j leads to $S = \Theta(N^2)$ possible moves), such that a complete neighborhood exploration takes $\mathcal{O}(B^2N^2)$ time. The amortized complexity per move evaluation, considering both preprocessing and effective evaluation, is thus $\mathcal{O}(\frac{N^2B^2}{S}) = \mathcal{O}(B^2)$. \square

5. Computational results

Computational experiments have been conducted on multiple benchmark instance sets. The first sets have been recently presented in Battarra et al. (2014). The authors considered instances proposed in the GVRP literature by Bektaş et al. (2011) (instance sets GVRP2 and GVRP3) and then generated larger instances by adapting the CVRP instances proposed by Golden et al. (1998) (instance set Golden) [using the method reported in Bektaş et al. \(2011\)](#). Among the instances proposed in Battarra et al. (2014), we have included in our benchmark set all Golden instances (200 up to 484 customers) and the most challenging ones among the GVRP2 and GVRP3 sets (the instances denoted as G and C in the GVRP literature, with 101 up to 262 customers). We have also generated an additional instance set with even larger problems (called hereafter Li), by adapting [the method of Bektaş et al. \(2011\)](#) on the instances originally proposed by Li et al. (2005). The latter set contains instances with up to 1200 customers. We decided to have clusters with average cardinality $\bar{\lambda} = 5$, leading to larger instances with 113 up to 241 clusters. A summary of the characteristics of our benchmark set is provided in Table 1. All sets of instances are available upon request and detailed result tables are displayed in the appendix (where the column *Exact* denotes the best upper bounds found in Battarra et al., 2014).

Table 1: Summary of benchmark set characteristics

Instance Set	Source	# Inst.	n	N
C	Bektaş et al. (2011)	2	101-200	34-100
G	Bektaş et al. (2011)	8	262-262	88-131
Golden	Battarra et al. (2014)	220	201-481	17-97
Li	New	12	560-1200	113 -241

An extensive calibration effort was spent in previous studies to find good and robust parameters for UHGS (Vidal et al., 2012) and ILS (Subramanian, 2012). We relied on this knowledge to obtain an initial parameter setting, and then scaled the parameters controlling algorithm termination to generate solutions for large-scale instances in comparable CPU time. As such, the

population-size parameters of UHGS are set to $(\mu^{\text{MIN}}, \mu^{\text{GEN}}) = (8, 8)$ and the termination criterion is $It_{\text{max}} = 400$. For ILS, the number of restarts has been set to $n_R = 50$ and the number of perturbations is $n_I = n + 5m$ as in Subramanian (2012). The choice of $n_I = 1000$ was adopted for ILS-Clu. All experiments have been conducted on a Xeon CPU with 3.07 GHz and 16 GB of RAM, running under Oracle Linux Server 6.4. Each algorithm was executed 10 times for each instance using a different random seed.

Table 2 summarizes the results obtained by the proposed hybrid metaheuristics. For each benchmark set, the number of instances “|Inst.|” is given, as well as the number of times “# BKS” the best known solution is found by ILS, ILS-Clu and UHGS, respectively. Columns 6-9 provide the average CPU time per instance in seconds. $UHGS_p$ also includes the CPU time dedicated to computing the cost of all intra-cluster Hamiltonian paths with Concorde. Columns 10-12 report the average percentage deviation from the best known solutions “Avg. % Dev.”. Note that the percentage deviation for a solution of value z from the best known solution value z_{BKS} is computed as $\frac{z - z_{\text{BKS}}}{z_{\text{BKS}}} \times 100$. The last row reports the overall number of best known solutions found by each method, the average CPU time and percentage average deviation.

From the experiments, it appears that UHGS is capable of finding most of the best known solutions (234 out of 242). In most cases, the average percentage gaps among the three methods is still small: ILS-Clu has an average deviation of 0.19% from the best known solutions and ILS has an average deviation of 0.13%. ILS is remarkably slower than the two other algorithms. The average CPU time for the large instances in the Li data set is 9548.6 seconds, versus 535.8, 345.3, 660.0 seconds for ILS-Clu, UHGS, and $UHGS_p$, respectively. Note that ILS performs about 0.06% better than ILS-Clu, but ILS-Clu is 15 times faster on average.

$UHGS_p$ is faster on average than ILS, even with the exhaustive search of all intra-cluster Hamiltonian paths using Concorde. This preprocessing phase is fast when the average cluster size is limited, but requires large CPU time when the cluster size increases, as in the case of the Golden instances. A heuristic evaluation of the cost of intra-cluster Hamiltonian paths could be a viable alternative. This is left as a research perspective. Finally, UHGS is faster than ILS-Clu for very large instances. ILS-Clu is on average faster on the G and C data sets, but slower on average on the Li set.

Table 2: Summary of results for the G, C, Golden and Li instances

Instance Set	Inst.	# BKS			Avg. Time (s)				Avg. % Dev.		
		ILS	ILS-Clu	UHGS	ILS	ILS-Clu	UHGS	$UHGS_p$	ILS	ILS-Clu	UHGS
G	2	0	1	2	127.6	53.5	150.2	165.2	0.64	0.22	0.00
C	8	6	8	7	26.0	17.8	27.1	35.1	0.19	0.04	0.05
Golden	220	127	87	213	698.8	53.9	53.7	854.9	0.11	0.19	0.01
Li	12	1	0	12	9548.6	535.8	345.3	660.0	0.34	0.21	0.00
Tot:	242	134	96	234	1110.7	76.6	68.1	812.4	0.13	0.19	0.01

Table 3: Summary of results for the Golden instance set grouped by instance size

n	# Opt.			Avg. Time (s)				Avg. % Dev.		
	ILS	ILS-Clu	UHGS	ILS	ILS-Clu	UHGS	UHGS _p	ILS	ILS-Clu	UHGS
201	11	11	11	81.18	19.92	14.57	2866.57	0	0	0
241	11	9	11	141.92	23.95	22.4	172.94	0	0.03	0
241	11	11	11	159.43	27.2	20.92	176.87	0	0	0
253	8	5	11	145.67	21.86	23.33	164.69	0.05	0.12	0
256	10	9	10	148.03	21.57	22.09	135.45	0.03	0.06	0.03
281	10	8	11	308.71	47.15	34.14	3848.32	0	0.03	0
301	10	11	11	309.59	37.9	30.75	191.26	0.02	0	0
321	3	2	11	442.25	47.27	47.57	243.39	0.08	0.07	0
321	7	2	11	333.56	34.29	38.12	152.78	0.12	0.25	0
324	6	4	11	336.87	31.55	43.82	175.73	0.19	0.31	0
361	3	1	11	816.36	73.97	66.94	2220.3	0.09	0.13	0
361	10	7	11	523.32	52.43	38.65	276.27	0.01	0.04	0
397	1	0	9	713.35	52.3	65.99	279.15	0.33	0.48	0.03
400	3	0	11	658.46	46.9	59.62	198	0.3	0.56	0
401	5	2	11	1115.99	85.16	82.91	1384.18	0.12	0.14	0
421	7	1	10	886.08	77.59	49.59	351.74	0.11	0.22	0.09
441	4	0	10	1573.62	101.69	97.09	1017.63	0.08	0.16	0
481	1	0	9	2336.64	130.52	137.95	2608.13	0.12	0.13	0.01
481	3	3	11	1344.21	74.34	84.3	246.31	0.15	0.39	0
484	3	1	11	1420.29	72.22	94.05	389.16	0.48	0.73	0
Tot:	127	87	213							

A more detailed comparison of the algorithms is displayed in Table 3 for the Golden instances. The large number of instances in this set allows for an analysis of the algorithms' performances by varying the number of customers and cluster size. The table reports aggregated results, obtained by averaging over instances with the same number of customers.

A correlation between the size of the instance and the performance of ILS can be observed; larger instances lead to larger gaps and higher CPU time. On the other hand, the performance of ILS-Clu is less dependent on instance size. For example, instances of group 12 with 484 customers are the most challenging for ILS-Clu with a 0.73% average deviation, but the deviation for instances of group 4, with size $n = 481$, is only 0.13% in average. A similar observation stands for UHGS, the most challenging instance groups being 4, 9, 14, and 20 with 481, 256, 397 and 421 customers, respectively.

Aggregating the Golden instances by average cluster size λ , as done in Table 4, leads to a further level of understanding of algorithms performance. All algorithms find solutions close to the best known when the average cluster size is large and therefore less clusters are present. The average CPU time of ILS does not depend on the average cluster size, whereas UHGS is consistently faster when large and few clusters are present. ILS-Clu attains its minimum CPU

Table 4: Summary of results for the Golden instances grouped by average cluster size

λ	# Opt.			Avg. Time (s)				Avg. Dev.		
	ILS	ILS-Clu	UHGS	ILS	ILS-Clu	UHGS	UHGS _p	ILS	ILS-Clu	UHGS
5	17	8	20	670.40	55.60	36.29	140.32	0.02	0.11	0.00
6	18	12	20	663.38	53.84	37.22	155.89	0.02	0.08	0.00
7	17	9	20	670.84	52.68	39.55	173.19	0.01	0.11	0.00
8	11	9	20	688.00	50.83	43.15	251.55	0.08	0.12	0.00
9	12	10	20	689.86	48.98	45.71	307.15	0.08	0.18	0.00
10	12	7	20	691.00	49.16	49.64	553.37	0.11	0.18	0.00
11	10	9	20	709.81	48.85	50.82	417.97	0.13	0.18	0.00
12	9	9	20	695.70	50.12	54.77	1025.66	0.13	0.18	0.00
13	8	5	20	725.73	53.12	69.04	916.16	0.18	0.29	0.00
14	7	5	17	699.89	59.99	73.52	2327.81	0.24	0.37	0.06
15	6	4	16	682.94	70.72	91.43	3135.31	0.23	0.31	0.03
Avg:	11.55	7.91	19.36	689.78	53.99	53.74	854.94	0.11	0.19	0.01

time when the average cluster size is approximately 9 customers. This is due to the fact that ILS-Clu performs both intra and inter-cluster LS moves; a balanced instance in terms of number and size of the clusters is a good compromise in terms of CPU time. Finally UHGS was capable of improving the best known solutions for five instances from Battarra et al. (2014). The values of these solutions are listed in Table 7.

6. Conclusions

This paper focused on the CluVRP, a generalization of the CVRP where customers are grouped into clusters. Three metaheuristics have been proposed, two of which are based on iterated local search, while the third is a hybrid genetic algorithm with a cluster-based solution representation. Efficient large neighborhood search procedures based on re-optimization techniques have been developed and integrated with the hybrid genetic search. The resulting three methods produce high quality solutions, and algorithms taking advantage of the cluster structure and large neighborhoods are remarkably faster. The hybrid genetic algorithm and large neighborhood search leads to solutions of higher quality than the two ILS based algorithms, but its pre-processing phase may become time consuming for instances with large clusters. Future work should consider heuristic preprocessing techniques to enhance CPU time, and other large neighborhoods strategies taking advantage of clusters.

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7. Detailed results

Table 5: Detailed results, Golden 1–4.

Inst.	n	N	m	Exact	ILS		ILS-Clu		Best	Avg.	UHGS	Preproc. (s)	Total Time(s)
					Best	Avg.	Best	Avg.					
Golden 1	241	17	4	4831	4831	4831.4	4831	4831.9	4831	4831	14.2	541	555.2
Golden 1	241	18	4	4847	4847	4849.6	4847	4848.4	4847	4847	17.1	151	168.1
Golden 1	241	19	4	4872	4872	4877.8	4872	4877.1	4872	4872	17.5	151	168.5
Golden 1	241	21	4	4889	4889	4896.2	4889	4891.2	4889	4889	17.6	147	164.6
Golden 1	241	22	4	4908	4908	4908.7	4908	4912.3	4908	4908	17.6	151	168.6
Golden 1	241	25	4	4899	4899	4905.9	4899	4899.9	4899	4899	20	144	164
Golden 1	241	27	4	4934	4934	4943	4934	4935.3	4934	4934	20.6	115	135.6
Golden 1	241	31	4	5050	5050	5051.5	5050	5050.2	5050	5050	21.6	90	111.6
Golden 1	241	35	4	5102	5102	5113.3	5102	5118	5102	5102	28.5	65	93.5
Golden 1	241	41	4	5097	5097	5108.4	5097	5113.3	5097	5097	31.9	55	86.9
Golden 1	241	49	3	5000	5000	5011.2	5000	5013.8	5000	5000	39.7	46	85.7
Golden 2	321	22	4	7716	7716	7719.5	7716	7722.2	7716	7716	31	368	399
Golden 2	321	23	4	7693	7693	7698.7	7693	7700.2	7693	7693	31.6	332	363.6
Golden 2	321	25	4	7668	7668	7675	7668	7677	7668	7668	32.8	305	337.8
Golden 2	321	27	4	7638	7638	7649.6	7638	7649.4	7638	7638	34.3	262	296.3
Golden 2	321	30	4	7617	7617	7635.3	7617	7644	7617	7617	37.6	172	209.6
Golden 2	321	33	4	7640	7640	7660.3	7640	7658	7640	7640	48.3	152	200.3
Golden 2	321	36	4	7643	7643	7650.5	7643	7660.4	7643	7643	49.1	167	216.1
Golden 2	321	41	4	7738	7738	7771.5	7738	7756.7	7738	7738	47.6	125	172.6
Golden 2	321	46	4	7861	7861	7888	7861	7884.7	7861	7861	55.6	99	154.6
Golden 2	321	54	4	7920	7920	7937.9	7920	7931.6	7920	7920	73	96	169
Golden 2	321	65	4	7892	7892	7910.8	7892	7906.3	7892	7892	82.4	76	158.4
Golden 3	401	27	4	10540	10540	10547.6	10540	10572.5	10540	10540	51.2	9678	9729.2
Golden 3	401	29	4	10504	10504	10513.5	10504	10520.9	10504	10504	58.2	2696	2754.2
Golden 3	401	31	4	10486	10486	10497.5	10486	10511.5	10486	10486	56	326	382
Golden 3	401	34	4	10465	10465	10492.7	10465	10497.6	10465	10465	59.6	372	431.6
Golden 3	401	37	4	10482	10482	10504.1	10482	10515.2	10482	10482	64.6	346	410.6
Golden 3	401	41	4	10501	10501	10533	10501	10544.2	10501	10501	76.1	290	366.1
Golden 3	401	45	4	10485	10485	10526.1	10485	10525.1	10485	10485	69.3	97	166.3
Golden 3	401	51	4	10583	10583	10637.3	10583	10639.6	10583	10583	83.6	153	236.6
Golden 3	401	58	4	10776	10776	10810.7	10776	10821.5	10776	10776	124.8	124	248.8
Golden 3	401	67	4	10797	10797	10843.2	10797	10844.8	10797	10797	121.5	123	244.5
Golden 3	401	81	4	10614	10614	10669.7	10614	10682	10614	10614	147.1	109	256.1
Golden 4	481	33	4	13598	13598	13614.3	13598	13628.5	13598	13598	87.4	7460	7547.4
Golden 4	481	35	4	13643	13643	13688.4	13643	13706.4	13643	13643	88.5	7434	7522.5
Golden 4	481	37	4	13520	13520	13575.9	13520	13569.3	13520	13520	89.1	1161	1250.1
Golden 4	481	41	4	13460	13460	13494.9	13460	13492.7	13460	13460	95.9	5438	5533.9
Golden 4	481	44	4	13568	13568	13604.9	13568	13601.5	13568	13568	121.1	1789	1910.1
Golden 4	481	49	4	13758	13758	13812.8	13758	13794.8	13758	13758	117.9	1659	1776.9
Golden 4	481	54	4	13760	13760	13829.5	13760	13816.6	13760	13760	121.2	113	234.2
Golden 4	481	61	4	13791	13791	13857.6	13791	13846.6	13791	13791	133.3	1635	1768.3
Golden 4	481	69	4	13966	13966	14034.8	13966	14022.8	13966	13966	195.6	198	393.6
Golden 4	481	81	4	13975	13975	14031.5	13975	14031.6	13975	13975	174	152	326
Golden 4	481	97	4	13775	13775	13857.5	13775	13849	13775	13775	293.5	133	426.5

Table 6: Detailed results, Golden 5–8.

Inst.	n	N	m	Exact			ILS			ILS-Clu			Best	Avg.	UHGS Time(s)	Preproc. (s)	Total Time(s)
				Best	Avg.	Time(s)	Best	Avg.	Time(s)	Best	Avg.	Time(s)					
Golden 5	201	14	4	7622	7622.5	74.5	7622	7640.8	22.5	7622	7622	8.3	7622	7651	7659.3		
Golden 5	201	15	3	7424	7443.6	94.2	7424	7427.6	23.9	7424	7424	12.2	7424	7568	7580.2		
Golden 5	201	16	3	7491	7491	94.3	7491	7491	25.1	7491	7491	13.3	7491	7758	7771.3		
Golden 5	201	17	3	7434	7445.8	90.6	7434	7439.9	21.5	7434	7434	13.3	7434	6893	6906.3		
Golden 5	201	19	4	7576	7576	71.2	7576	7576	16.9	7576	7576	12.8	7576	498	510.8		
Golden 5	201	21	4	7596	7596.3	76.7	7596	7596.3	16	7596	7596	12.8	7596	209	221.8		
Golden 5	201	23	4	7643	7643	78.7	7643	7643	16.9	7643	7643	14.6	7643	151	165.6		
Golden 5	201	26	4	7560	7560	78.3	7560	7560	18	7560	7560	14.9	7560	221	235.9		
Golden 5	201	29	4	7410	7410	75.3	7410	7410	17.1	7410	7410	16.5	7410	219	235.5		
Golden 5	201	34	4	7429	7429	78.6	7429	7429	20.1	7429	7429	19.4	7429	164	183.4		
Golden 5	201	41	4	7241	7243	80.5	7241	7241	21	7241	7241	22.1	7241	40	62.1		
Golden 6	281	19	3	8624	8629.2	353.4	8624	8624	61.5	8624	8624	29.1	8624	18971	19000.1		
Golden 6	281	21	3	8628	8629.7	371.1	8628	8632.4	59.8	8628	8628	27.7	8628	20337	20364.7		
Golden 6	281	22	3	8646	8648	378.7	8646	8650.1	59.6	8646	8646	29.9	8646	789	818.9		
Golden 6	281	24	4	8853	8868.5	281	8853	8869.8	41	8853	8853	26	8853	525	551		
Golden 6	281	26	4	8910	8916.6	271.9	8910	8921.1	39.2	8910	8910	28	8910	465	493		
Golden 6	281	29	4	8936	8962.8	302.7	8936	8963.7	40.5	8936	8936	34.3	8936	257	291.3		
Golden 6	281	32	4	8891	8905.8	282.7	8891	8899.9	39	8891	8891	30.6	8891	92	122.6		
Golden 6	281	36	4	8969	8970.4	284.7	8969	8971.6	38.8	8969	8969	32.3	8969	160	192.3		
Golden 6	281	41	4	9028	9039.8	284.6	9028	9043.2	40.4	9028	9028	43	9028	150	193		
Golden 6	281	47	4	8923	8924.2	282.8	8923	8923	46.3	8923	8923	39.6	8923	134	173.6		
Golden 6	281	57	4	9028	9052.6	302.2	9031	9052.1	52.6	9028	9028.7	54.9	9028	76	130.9		
Golden 7	361	25	3	9904	9931	1038.6	9922	9947.9	90.2	9904	9904	57.7	9904	12354	12411.7		
Golden 7	361	26	3	9888	9907	951.4	9905	9924.4	84.4	9888	9888	57.4	9888	3017	3074.4		
Golden 7	361	28	3	9917	9944	946.7	9922	9949.9	80	9917	9917	61.7	9917	3092	3153.7		
Golden 7	361	31	4	10021	10035	10048.9	10025	10054.7	66.1	10021	10021	48.5	10021	2626	2674.5		
Golden 7	361	33	4	10029	10040	10057.2	10042	10059.8	62.6	10029	10029	51.8	10029	980	1031.8		
Golden 7	361	37	4	10131	10141	10150	10134	10148.1	65.7	10131	10131	62	10131	1005	1067		
Golden 7	361	41	4	10052	10071.3	736.5	10052	10067.6	67.1	10052	10052	58.1	10052	116	174.1		
Golden 7	361	46	4	10080	10100	744.7	10094	10113.1	64.9	10080	10080	62.5	10080	201	263.5		
Golden 7	361	52	4	10095	10096	804.7	10116	10130.6	65.3	10095	10095	83	10095	108	191		
Golden 7	361	61	4	10096	10115	823.9	10128	10139.5	77.4	10096	10096	84.5	10096	93	177.5		
Golden 7	361	73	4	10014	10038	10051.9	10034	10057	89.9	10014	10014	109.1	10014	95	204.1		
Golden 8	441	30	4	10866	10885.5	1650.7	10883	10897.3	102.7	10866	10866	67.1	10866	1004	1071.1		
Golden 8	441	32	4	10831	10845.8	1700.1	10845	10866.9	104.2	10831	10831	67.3	10831	1003	1070.3		
Golden 8	441	34	4	10847	10849	1565.7	10852	10892	104	10847	10847	65.9	10847	970	1035.9		
Golden 8	441	37	4	10859	10871	1530.1	10883	10893.7	100.2	10859	10859	76.2	10859	829	905.2		
Golden 8	441	41	4	10934	10934	1660.8	10945	10959	94.5	10934	10934	81.8	10934	792	873.8		
Golden 8	441	45	4	10960	10966.5	1503.3	10969	10987.8	88.9	10960	10960	82.4	10960	4836	4918.4		
Golden 8	441	49	4	11042	11064	1568.9	11060	11083.4	83.6	11042	11042	95.8	11042	127	222.8		
Golden 8	441	56	4	11194	11211	1678.1	11210	11251.9	87.7	11194	11194	92.6	11194	166	258.6		
Golden 8	441	63	4	11252	11267	1563.2	11267	11309	100.9	11252	11252	127.4	11252	155	282.4		
Golden 8	441	74	4	11321	11347	1394.3	11357	11371	114.4	11321	11321	162.7	11321	132	294.7		
Golden 8	441	89	4	11209	11215	1494.7	11243	11271.8	137.7			148.8		112	260.8		

Table 7: Detailed results, Golden 9–12.

Inst.	n	N	m	Exact	ILS		ILS-Clu		Best	Avg. Time(s)	UHGS			
					Best	Avg. Time(s)	Best	Avg. Time(s)			Avg. Time(s)	Preproc. (s)	Total Time(s)	
Golden 9	256	18	4	300	300	142.3	300	300	22.7	300	300	15	164	179
Golden 9	256	19	4	299	299.3	135.3	299	299.4	22	299	299	16.1	157	173.1
Golden 9	256	20	4	296	296	144.7	296	296.3	21.9	296	296	17.6	133	150.6
Golden 9	256	22	4	290	291	141.9	290	291.1	21.3	290	290	18.2	138	156.2
Golden 9	256	24	4	290	290.9	146.8	290	291.6	20.6	290	290	19.3	113	132.3
Golden 9	256	26	4	288	288.5	149.7	288	290.2	18.9	288	288	21.5	92	113.5
Golden 9	256	29	4	292	293.7	146.4	292	294.8	19.2	292	292	23.6	211	234.6
Golden 9	256	32	4	297	298	148.9	297	298.3	20.9	297	297	22.5	79	101.5
Golden 9	256	37	4	294	294.7	149.5	294	294.5	21.1	294	294	25.9	68	93.9
Golden 9	256	43	4	295	296.2	156.8	296	296.9	22.8	295	295.8	31.6	59	90.6
Golden 9	256	52	4	296	297	166	297	298.2	26	297	297	31.6	33	64.6
Golden 10	324	22	4	367	368.6	331.3	369	369.9	31.2	367	367	30.3	230	260.3
Golden 10	324	24	4	361	362.5	319.4	361	362.2	28.7	361	361	31.6	175	206.6
Golden 10	324	25	4	359	360.4	322.4	359	360	29.3	359	359	31.9	173	204.9
Golden 10	324	27	4	361	362.1	329.9	361	362.6	29.9	361	361	33.4	168	201.4
Golden 10	324	30	4	367	368.8	350	368	369	30.6	367	367.1	40.6	145	185.6
Golden 10	324	33	4	373	375	376.6	376	377.2	31.1	373	373.2	37.7	125	162.7
Golden 10	324	36	4	385	387	387.7	387	388.9	29.3	385	385.2	42.1	144	186.1
Golden 10	324	41	4	400	401	402.4	400	401.9	29.7	400	400.2	51.8	83	134.8
Golden 10	324	47	4	398	399	400.1	399	400.1	31.6	398	398.1	52.7	74	126.7
Golden 10	324	54	4	393	395	396.9	394	395.3	34.9	393	393.6	54.5	72	126.5
Golden 10	324	65	4	387	391.7	355.7	390	392	40.9	387	387.9	75.5	62	137.5
Golden 11	400	27	5	457	458.1	596.1	458	459	42.4	457	457	38.3	238	313.5
Golden 11	400	29	5	455	457.8	586.6	456	458.8	42.7	455	455	44.5	222	222
Golden 11	400	31	5	455	457.1	615.1	457	459.1	42.2	455	455.4	49.8	217	217
Golden 11	400	34	5	455	456	648.9	457	458.7	40.8	455	455.2	49.6	188	188
Golden 11	400	37	5	459	460	693.7	461	462.6	41	459	459	49.9	163	163
Golden 11	400	40	5	461	462	677.7	463	463.5	41.3	461	461	48.4	142	180.3
Golden 11	400	45	5	462	464	636.4	465	636.4	43.5	462	462.1	52.7	270	314.5
Golden 11	400	50	5	458	459	677.3	460	461.2	45.9	458	458.5	57.8	123	172.8
Golden 11	400	58	5	456	458	701.1	460	462	49.6	456	456.4	80.3	105	154.6
Golden 11	400	67	5	454	457	733.8	458	460.5	59.7	454	454.6	85.7	91	140.9
Golden 11	400	80	5	451	455	676.3	456	458.6	67	451	451.7	98.8	63	111.4
Golden 12	484	33	5	535	538.1	1338.3	537	538.4	63	535	535	80.3	270	322.7
Golden 12	484	35	5	537	538.4	1367.3	537	539.7	60.4	537	537	62.6	255	312.8
Golden 12	484	38	5	535	540	1327.9	537	540.5	61.9	535	535	70.3	231	311.3
Golden 12	484	41	5	537	539	1516.8	538	542.7	63.9	537	537	70.9	200	285.7
Golden 12	484	44	5	535	537	1406.3	540	542.2	62.4	535	535.8	87.4	194	292.8
Golden 12	484	49	5	533	537	1443	538	540.9	68.9	533	533.4	104	192	272.3
Golden 12	484	54	5	535	537	1679.6	539	542.9	69.5	535	535.3	92.1	1656	1718.6
Golden 12	484	61	5	538	541.3	1453.1	539	544.8	72.7	535	535.2	89.4	152	222.3
Golden 12	484	70	5	546	538	1519.4	540	543.6	80.5	533	533.5	104.4	131	201.9
Golden 12	484	81	5	546	541	1296.2	544	547.5	85.4	535	536	133.7	93	180.4
Golden 12	484	97	5	560	548	1275.5	548	554.5	105.9	544	544.7	139.6	56	160

Table 8: Detailed results, Golden 13–16.

Inst.	n	N	m	Exact	ILS		ILS-Clu		Best	Avg.		Best	UHGS		Preproc. (s)	Total Time(s)
					Best	Avg.	Avg.	Time(s)		Avg.	Time(s)					
Golden 13	253	17	4	552	552	144.7	553.8	21	552	552	552	552	15.8	159	251.1	
Golden 13	253	19	4	549	549.1	138.4	551.3	20.7	551	549	549	549	16.4	128	217.4	
Golden 13	253	20	4	548	548.3	140.9	549.5	19.7	548	548	548	548	17.9	127	231.4	
Golden 13	253	22	4	548	548.4	141.2	549.5	20.5	549	548	548	548	20.3	110	243.7	
Golden 13	253	23	4	548	548.5	138.5	549.1	20.1	548	548	548	548	19.6	107	246.6	
Golden 13	253	26	4	542	542.1	146.6	542.6	20.2	542	542	542	542	20.4	94	109.8	
Golden 13	253	29	4	540	540.3	147.7	540.9	21.3	540	540	540	540	21.8	218	234.4	
Golden 13	253	32	4	543	543.7	145.7	544.9	21.3	543	543	543	543	23.7	72	89.9	
Golden 13	253	37	4	545	546	147.9	547.4	21.7	546	548.8	545.2	545	33.7	54	74.3	
Golden 13	253	43	4	553	554	146.4	555.1	25.3	554	555.6	553	553	29.2	44	63.6	
Golden 13	253	51	4	560	561	164.9	562.1	28.6	561	563	560.4	560	37.9	29	49.4	
Golden 14	321	22	4	692	692.8	320.7	693	695	35.4	692	692	692	25.6	214	235.8	
Golden 14	321	23	4	688	688.3	330.6	688.7	32.2	688	688	688	688	27.3	181	204.7	
Golden 14	321	25	4	678	678.1	317.3	679	680	31.2	678	678	678	29	169	202.7	
Golden 14	321	27	4	676	677.6	317.3	678.6	31.1	676	676	676	676	29.3	147	176.2	
Golden 14	321	30	4	678	680	319.3	680.5	30.5	682	682.3	678	678	32.2	128	165.9	
Golden 14	321	33	4	682	683.6	341.5	684	685.3	31	682	682	682	32.9	118	143.6	
Golden 14	321	36	4	687	688.3	349.3	688.3	33	688	689.5	687	687	33.7	163	190.3	
Golden 14	321	41	4	690	691	339.7	691	33.7	691	692.9	690	690	50.2	83	112	
Golden 14	321	46	4	694	697	352	698	34.1	697	699	694	694	48.5	65	94.3	
Golden 14	321	54	4	699	701	339	703	38.2	703	704.2	699	699	52.7	53	85.2	
Golden 14	321	65	4	703	706.3	342.7	704	705.9	47	703	703	703	58	37	69.9	
Golden 15	397	27	4	842	844.2	728.5	844	846.3	54.9	842	842	842	47.5	244	277.7	
Golden 15	397	29	4	843	846.4	736.6	845	849.2	52.2	843	843.4	843	52.5	218	268.2	
Golden 15	397	31	4	837	839	742.7	841	843.3	48.5	837	837.1	837	49.3	203	251.5	
Golden 15	397	34	4	838	842	755.2	844	846.6	48.2	838	838.5	838	72.4	169	221.7	
Golden 15	397	37	4	845	848	739.6	848	853.5	48.5	845	845.2	845	51.6	152	210	
Golden 15	397	40	4	849	851.9	744.5	850	853	50	849	849.2	849	65.3	141	188.5	
Golden 15	397	45	5	853	856	655.4	855	858.4	45.5	853	853.1	853	58.5	1033	1085.5	
Golden 15	397	50	5	851	854	710.7	857	859	48.8	851	851.8	851	68.1	111	160.3	
Golden 15	397	57	5	850	854	700.3	856	858.3	50.6	850	850.4	850	83.5	94	166.4	
Golden 15	397	67	5	855	857	685.4	861	862.9	57.2	857	857.3	857	82.9	73	124.6	
Golden 15	397	80	5	857	863	648.1	862	864.2	70.9	858	859.6	858	94.3	51	116.3	
Golden 16	481	33	5	1030	1030	1268.8	1030	1031.2	67.7	1030	1030	1030	54.7	296	354.5	
Golden 16	481	35	5	1028	1029.4	1238	1028	1030.6	65.5	1028	1028	1028	59.5	251	319.1	
Golden 16	481	37	5	1028	1029	1288.6	1028	1029.7	63.1	1028	1028	1028	60.9	238	321.5	
Golden 16	481	41	5	1032	1033	1329.9	1033	1034.6	64.8	1032	1032	1032	60.1	199	281.9	
Golden 16	481	44	5	1028	1029	1031.2	1032	1033.3	64.6	1028	1028	1028	63.8	179	273.3	
Golden 16	481	49	5	1031	1033	1034.6	1034	1036	65.4	1031	1031	1031	71.2	161	215.7	
Golden 16	481	54	5	1022	1024	1505.4	1027	1028.8	69.2	1022	1022	1022	83.9	214	273.5	
Golden 16	481	61	5	1013	1015	1018.7	1015	1018.7	74	1013	1013.8	1013	94.7	143	203.9	
Golden 16	481	69	5	1012	1015	1017.7	1015	1017.7	79.4	1012	1012.3	1012	114.5	120	180.1	
Golden 16	481	81	5	1018	1019	1023.7	1023	1026.5	93.2	1018	1018	1018	105.3	94	157.8	
Golden 16	481	97	5	1018	1023	1027.1	1023	1029.9	110.9	1018	1020	1018	158.8	57	128.2	

Table 10: Detailed results, GVRP2

Inst.	n		m	Exact	ILS			ILS-Clu			UHGS				
					Best	Avg.	Time(s)	Best	Avg.	Time(s)	Best	Avg.	Time(s)	Preproc. (s)	Total Time(s)
G	262	131	12	—	3736	3749.6	127.1	3709	3718	66.8	3693	3711	230.9	9	239.9
C	101	51	5	642	642	642	5.1	642	642	8.5	642	642	9.2	3	12.2
C	121	61	4	807	807	809.2	15.7	807	807	16.3	807	808.3	22.8	4	26.8
C	151	76	6	816	816	817.5	22.2	816	816	21.8	816	817.6	27.4	4	31.4
C	200	100	8	994	965	971.6	54.3	955	961.4	35.6	960	964.2	89.8	8	97.8

Table 11: Detailed results, GVRP3

Inst.	n	N	m	Exact	ILS			ILS-Clu			UHGS				
					Best	Avg.	Time(s)	Best	Avg.	Time(s)	Best	Avg.	Time(s)	Preproc. (s)	Total Time(s)
G	262	88	9	3596	3294	3305.9	128	3290	3291.8	40.2	3290	3293.2	69.6	21	90.6
C	101	34	4	607	607	607	7.4	607	607	7.1	607	607	7.3	8	15.3
C	121	41	3	691	694	695.4	14.3	691	691	12.6	691	691	17.8	9	26.8
C	151	51	4	804	804	804	29.5	804	804	16.2	804	804	18.4	12	30.4
C	200	67	6	908	908	908	59.5	908	908	24.4	908	908	24.1	16	40.1

Table 12: Detailed results, Li

Inst.	n	N	m	ILS			ILS-Clu			UHGS				
				Best	Avg.	Time(s)	Best	Avg.	Time(s)	Best	Avg.	Time(s)	Preproc. (s)	Total Time(s)
Li	560	113	39	27962	28079.7	714.6	27970	27997.1	122.7	27962	27963	83.9	200	283.9
Li	600	121	62	29087	29111.8	783.6	29063	29077.6	123	29051	29055.6	94.8	196	290.8
Li	640	129	10	21368	21515.2	2930.2	21365	21435.3	240.8	21243	21330.9	254.6	265	519.6
Li	720	145	11	24777	24834.4	4444.1	24578	24664.5	330	24486	24557.4	315.5	327	642.5
Li	760	153	78	35190	35206.7	1669.5	35175	35182.9	242.6	35166	35169.6	143.2	284	427.2
Li	800	161	11	27350	27482.7	7422.5	27284	27362.8	457.8	27238	27307.7	363.8	263	626.8
Li	840	169	86	37878	37895	1881.6	37860	37871.3	298.3	37859	37871.4	222.8	302	524.8
Li	880	177	11	30637	30766.5	10824.4	30568	30665.7	571.3	30483	30580.2	444.7	402	846.7
Li	960	193	11	32873	32958	14578.7	32763	32888.3	742.6	32656	32717.1	515.6	290	805.6
Li	1040	209	11	35991	36083	20634.3	35976	36044.6	843.5	35885	35919.1	554.6	440	994.6
Li	1120	225	11	38690	38749	22831.5	38727	38785.8	1187.2	38669	38687.8	566.9	397	963.9
Li	1200	241	11	41534	41621.2	25868.3	41563	41622.9	1269.8	41461	41490.7	582.8	411	993.8